Concurrency Theory

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Lecture 12: Modelling Mutual-Exclusion Algorithms & Value-Passing CCS

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https://proglang.github.io/teaching/25ws/ct.html

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Peterson's Mutual Exclusion Algorithm

- **Goal:** ensuring exclusive access to non-shared resources
- Here: two competing processes P_1, P_2 and shared variables
  - b_1, b_2 (Boolean, initially false) – b_i indicates that P_i wants to enter critical section
  - k (in $\{1, 2\}$, arbitrary initial value) – index of prioritised process ("turn variable")
- P_i uses local variable $j := 2 - i$ (index of other process)

Algorithm 12.1 (Peterson's algorithm for P_i)

while true do
  "non-critical section";
  $b_i :=$ true;
  $k := j$;
  await $\neg b_j \lor k = i$;
  "critical section";
  $b_i :=$ false;
end

Representing Shared Variables in CCS

- Not directly expressible in CCS (communication by handshaking)
- Idea: consider variables as processes that communicate with environment by processing read/write requests

Example 12.2 (Shared variables in Peterson's algorithm)

- Encoding of b_1 with two (process) states B_{1t} (value tt) and B_{1f} (ff)
- Read access along ports $b1rt$ (in state B_{1t}) and $b1rf$ (in state B_{1f})
- Write access along ports $b1wt$ and $b1wf$ (in both states)
- Possible behaviours:

$$\begin{aligned} B_{1f} &= \overline{b1rf}.B_{1f} + b1wf.B_{1f} + b1wt.B_{1t} \\ B_{1t} &= \overline{b1rt}.B_{1t} + b1wf.B_{1f} + b1wt.B_{1t} \end{aligned}$$

- Similarly for b_2 and k:

$$\begin{aligned} B_{2f} &= \overline{b2rf}.B_{2f} + b2wf.B_{2f} + b2wt.B_{2t} \\ B_{2t} &= \overline{b2rt}.B_{2t} + b2wf.B_{2f} + b2wt.B_{2t} \end{aligned}$$

$$K_1 = \overline{kr1}.K_1 + kw1.K_1 + kw2.K_2$$

Modelling the Processes in CCS

Assumption: P_i cannot fail or terminate within critical section

Peterson's algorithm

while true do
  "non-critical section";
  $b_i :=$ true;
  $k := j$;
  await $\neg b_j \vee k = i$;
  "critical section";
  $b_i :=$ false;
end

CCS representation

$$\begin{aligned}
P_1 &= \overline{b1wt}.\overline{kw2}.P_{11} \\
P_{11} &= b2rf.P_{12} + \\
&\quad\; b2rt.(kr1.P_{12} + kr2.P_{11}) \\
P_{12} &= enter1.exit1.\overline{b1wf}.P_1
\end{aligned}$$

$$\begin{aligned}
P_2 &= \overline{b2wt}.\overline{kw1}.P_{21} \\
P_{21} &= b1rf.P_{22} + \\
&\quad\; b1rt.(kr1.P_{21} + kr2.P_{22}) \\
P_{22} &= enter2.exit2.\overline{b2wf}.P_2
\end{aligned}$$

$$Peterson = (P_1 \parallel P_2 \parallel B_{1f} \parallel B_{2f} \parallel K_1) \setminus L$$

for $L = \{b1rf, b1rt, b1wf, b1wt, b2rf, b2rt, b2wf, b2wt, kr1, kr2, kw1, kw2\}$

Specifying the Algorithm in CAAL

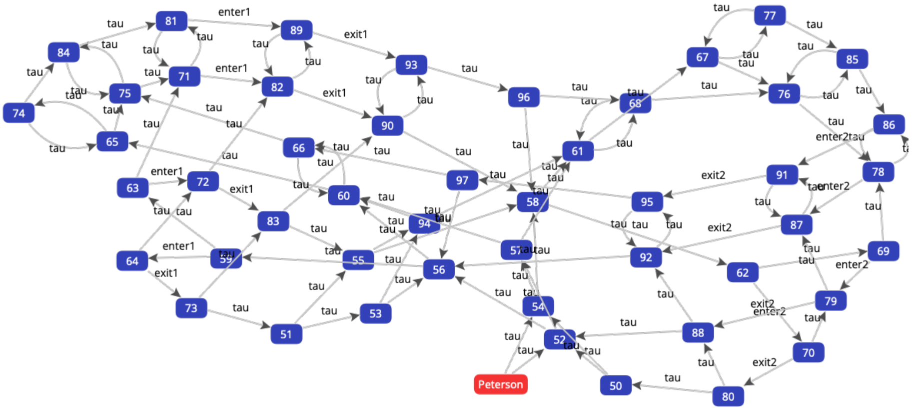
CAAL Project Edit Explore Verify Games About Syntax

MutEx

Parse CCS TCCS 14

```
*
* Peterson's algorithm for mutual exclusion (Lecture 12)
*

B1f = 'b1rf.B1f + b1wf.B1f + b1wt.B1t;
* B1f = 'b1rf.B1f + b1wf.B1f + b1wt.enabled1.B1t;
B1t = 'b1rt.B1t + b1wf.B1f + b1wt.B1t;

B2f = 'b2rf.B2f + b2wf.B2f + b2wt.B2t;
* B2f = 'b2rf.B2f + b2wf.B2f + b2wt.enabled2.B2t;
B2t = 'b2rt.B2t + b2wf.B2f + b2wt.B2t;

K1 = 'kr1.K1 + kw1.K1 + kw2.K2;
K2 = 'kr2.K2 + kw1.K1 + kw2.K2;

P1 = 'b1wt.'kw2.P11;
P11 = b2rf.P12 + b2rt.(kr2.P11 + kr1.P12);
P12 = enter1.exit1.'b1wf.P1;

P2 = 'b2wt.'kw1.P21;
P21 = b1rf.P22 + b1rt.(kr1.P21 + kr2.P22);
P22 = enter2.exit2.'b2wf.P2;

set L = {b1rf, b2rf, b1rt, b2rt, b1wf, b2wf, b1wt, b2wt, kr1, kr2, kw1, kw2};
Peterson = (P1 | P2 | B1f | B2f | K1) \ L;

MutExCCS = enter1.exit1.MutExCCS + enter2.exit2.MutExCCS;
```

Obtaining the LTS Using CAAL



The Mutual Exclusion Property

- **Done:** Formal description of Peterson's algorithm
- **To do:** Analysing its behaviour (manually or with tool support)
- **Question:** What does "ensuring mutual exclusion" formally mean?

Mutual exclusion

At no point in the execution of the algorithm, processes P_1 and P_2 will both be in their critical section.

Equivalently:
It is always the case that either P_1 or P_2 or both are not in their critical section.

Model Checking Mutual Exclusion in HML

Mutual exclusion

It is always the case that either P_1 or P_2 or both are not in their critical section.

Observations:

- Mutual exclusion is an invariance property ("always").
- P_i is in its critical section iff action *exit i* is enabled.

Mutual exclusion in HML

$$Peterson \models MutExHML$$

$$MutExHML := Inv(NotBoth)$$

$$Inv(F) \stackrel{max}{=} F \wedge [Act]Inv(F) \qquad \text{(cf. Theorem 11.1)}$$

$$NotBoth := [exit1]\mathrm{ff} \vee [exit2]\mathrm{ff}$$

Absence of Deadlocks

Absence of deadlocks

It is always the case that the system can progress, i.e., perform any action.

Absence of deadlocks in HML

$$Peterson \models NoDeadlock$$

$$\begin{aligned} NoDeadlock &:= Inv(CanProgress) \\ Inv(F) &\stackrel{max}{=} F \wedge [Act]Inv(F) \\ CanProgress &:= \langle Act \rangle \mathrm{tt} \end{aligned}$$

Possibility of Livelocks

Possibility of livelocks

It is possible for the system to reach a state which has a livelock (i.e., an infinite sequence of internal steps).

Possibility of livelocks in HML (cf. Example 11.11)

$$Peterson \models HasLivelock$$

$$\begin{aligned} HasLivelock &:= Pos(Livelock) \\ Pos(F) &\stackrel{min}{=} F \vee \langle Act \rangle Pos(F) \qquad \text{(cf. Theorem 11.2)} \\ Livelock &\stackrel{max}{=} \langle \tau \rangle Livelock \end{aligned}$$

Verification by Bisimulation Checking

- **Goal:** express desired behaviour of algorithm as an "abstract" CCS process
- Intuitively:
  (1) Initially, either P_1 or P_2 can enter its critical section.
  (2) Afterwards, the other process cannot enter the critical section before the first has left.

Mutual exclusion in CCS

$$\mathit{MutExCCS} = \mathit{enter1}.\mathit{exit1}.\mathit{MutExCCS} + \mathit{enter2}.\mathit{exit2}.\mathit{MutExCCS}$$

- Weak bisimilarity (Definition 5.10) does *not* hold as *Peterson* satisfies additional fairness constraints (prioritisation via variable k — distinguishing formula $\langle\langle\tau\rangle\rangle[[\mathit{enter2}]]\mathrm{ff}$):
$$\mathit{Peterson} \not\approx \mathit{MutExCCS}.$$
- However, *Peterson* and *MutExCCS* are weakly simulation equivalent:
$$\mathit{Peterson} \precsim \mathit{MutExCCS} \qquad \text{and} \qquad \mathit{MutExCCS} \precsim \mathit{Peterson}$$
where $P \precsim Q$ denotes that Q weakly simulates P, i.e., can respond to every $\xrightarrow{\alpha}$-step of P by performing a $\xRightarrow{\alpha}$-step (cf. Definition 5.3 of strong

Verification by Testing I

Approach:

- Make mutual exclusion algorithm interact with "monitor" process that observes its behaviour.
- Report error if and when undesired behaviour is detected.

The monitor process

$$
\begin{aligned}
\mathit{MutExTest} &:= \overline{\mathit{enter1}}.\mathit{MutExTest}_1 + \overline{\mathit{enter2}}.\mathit{MutExTest}_2 \\
\mathit{MutExTest}_1 &:= \overline{\mathit{exit1}}.\mathit{MutExTest} + \overline{\mathit{enter2}}.\overline{\mathit{bad}}.\mathrm{nil} \\
\mathit{MutExTest}_2 &:= \overline{\mathit{exit2}}.\mathit{MutExTest} + \overline{\mathit{enter1}}.\overline{\mathit{bad}}.\mathrm{nil}
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Test} &:= (\mathit{Peterson} \parallel \mathit{MutExTest}) \setminus L' \\
L' &:= \{\mathit{enter1}, \mathit{enter2}, \mathit{exit1}, \mathit{exit2}\}
\end{aligned}
$$

Verification by Testing II

Lemma 12.2

Let $P \in Prc$ be a process whose only visible actions are contained in L'. Then

$$Test \stackrel{\overline{bad}}{\Longrightarrow} \quad \text{iff} \quad \text{either } P \stackrel{\sigma}{\Longrightarrow} \stackrel{enter1}{\Longrightarrow} \stackrel{enter2}{\Longrightarrow} \text{ or } P \stackrel{\sigma}{\Longrightarrow} \stackrel{enter2}{\Longrightarrow} \stackrel{enter1}{\Longrightarrow}$$

for some sequence of actions $\sigma \in (enter1\ exit1 \mid enter2\ exit2)^*$.

Proof.

see Luca Aceto, Anna Ingólfsdóttir, Kim Guldstrand Larsen and Jiří Srba: *Reactive Systems: Modelling, Specification and Verification*, Cambridge University Press, 2007, Proposition 7.2 □

Absence of bad transitions

$$Test \models NoBadTransition$$

$$NoBadTransition := Inv([\overline{bad}]\mathrm{ff})$$

$$Inv(F) \stackrel{max}{=} F \wedge [Act]Inv(F)$$

Value-Passing CCS

- **So far:** pure CCS
  - communication = mere synchronisation
  - no (explicit) exchange of data
- **But:** processes usually do pass around data

⇒ Value-passing CCS

- Introduced in Robin Milner: *Communication and Concurrency*, Prentice-Hall, 1989
- Assumption (for simplicity): only integers as data type

Example 12.3 (One-place buffer with data; cf. Example 2.5)

One-place buffer that outputs successor of stored value:

$$B = in(x).B'(x)$$
$$B'(x) = \overline{out}(x+1).B$$

Syntax of Value-Passing CCS I

Definition 12.4 (Syntax of value-passing CCS)

- Let A, $\overline{A}$, Pid (ranked) as in Definition 2.1.
- Let e and b be integer and Boolean expressions, resp., built from integer variables $x, y, \ldots$
- The set Prc^+ of value-passing process expressions is defined by:

$$
\begin{array}{rlll}
P ::= & \mathrm{nil} & & \text{(inaction)} \\
\mid & a(x).P & & \text{(input prefixing)} \\
\mid & \overline{a}(e).P & & \text{(output prefixing)} \\
\mid & \tau.P & & \text{(τ prefixing)} \\
\mid & P_1 + P_2 & & \text{(choice)} \\
\mid & P_1 \parallel P_2 & & \text{(parallel composition)} \\
\mid & P \setminus L & & \text{(restriction)} \\
\mid & P[f] & & \text{(relabelling)} \\
\mid & \text{if } b \text{ then } P & & \text{(conditional)} \\
\mid & C(e_1, \ldots, e_n) & & \text{(process call)}
\end{array}
$$

Syntax of Value-Passing CCS II

Definition 12.4 (Syntax of value-passing CCS; continued)

A value-passing process definition is an equation system of the form

$$(C_i(x_1, \ldots, x_{n_i}) = P_i \mid 1 \leq i \leq k)$$

where

- $k \geq 1$,
- $C_i \in Pid$ of rank n_i (pairwise distinct),
- $P_i \in Prc^+$ (with process identifiers from $\{C_1, \ldots, C_k\}$), and
- all occurrences of an integer variable y in each P_i are bound, i.e., $y \in \{x_1, \ldots, x_{n_i}\}$ or y is in the scope of an input prefix of the form $a(y)$ (to ensure well-definedness of values).

Example 12.5

(1) $C(x) = \overline{a}(x+1).b(y).C(y)$ is allowed

(2) $C(x) = \overline{a}(x+1).\overline{a}(y+2).\mathrm{nil}$ is disallowed as y is not bound

Semantics of Value-Passing CCS I

Definition 12.6 (Semantics of value-passing CCS)

A value-passing process definition $(C_i(x_1, \ldots, x_{n_i}) = P_i \mid 1 \leq i \leq k)$ determines the LTS $(Prc^+, Act, \longrightarrow)$ with $Act := (A \cup \overline{A}) \times \mathbb{Z} \cup \{\tau\}$ whose transitions can be inferred from the following rules ($P, P', Q, Q' \in Prc^+$, $a \in A$, x_i integer variables, e_i/b integer/Boolean expressions, $z \in \mathbb{Z}$, $\alpha \in Act$, $\lambda \in (A \cup \overline{A}) \times \mathbb{Z}$):

$$\text{(In)}\frac{}{a(x).P \xrightarrow{a(z)} P[z/x]} \qquad \text{(Out)}\frac{(z \text{ value of } e)}{\overline{a}(e).P \xrightarrow{\overline{a}(z)} P} \qquad \text{(Tau)}\frac{}{\tau.P \xrightarrow{\tau} P}$$

$$(\text{Sum}_1)\frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'} \qquad (\text{Sum}_2)\frac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'}$$

$$(\text{Par}_1)\frac{P \xrightarrow{\alpha} P'}{P \parallel Q \xrightarrow{\alpha} P' \parallel Q} \qquad (\text{Par}_2)\frac{Q \xrightarrow{\alpha} Q'}{P \parallel Q \xrightarrow{\alpha} P \parallel Q'} \qquad \text{(Com)}\frac{P \xrightarrow{\lambda} P' \quad Q \xrightarrow{\overline{\lambda}} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'}$$

Definition 12.6 (Semantics of value-passing CCS; continued)

$$\text{(Rel)}\ \frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(\alpha)} P'[f]} \qquad \text{(Res)}\ \frac{P \xrightarrow{\alpha} P' \quad (\alpha \notin (L \cup \overline{L}) \times \mathbb{Z})}{P \setminus L \xrightarrow{\alpha} P' \setminus L}$$

$$\text{(If)}\ \frac{P \xrightarrow{\alpha} P' \quad (b \text{ true})}{\text{if } b \text{ then } P \xrightarrow{\alpha} P'} \qquad \text{(Call)}\ \frac{\begin{array}{c} P[z_1/x_1, \ldots, z_n/x_n] \xrightarrow{\alpha} P' \\ (C(x_1, \ldots, x_n) = P, z_i \text{ value of } e_i) \end{array}}{C(e_1, \ldots, e_n) \xrightarrow{\alpha} P'}$$

Remarks:

- $P[z_1/x_1, \ldots, z_n/x_n]$ denotes the substitution of each free occurrence of x_i by z_i $(1 \leq i \leq n)$.
- Operations on actions ignore values:
  $\overline{a(z)} := \overline{a}(z) \quad \overline{\overline{a}(z)} := a(z) \quad f(a(z)) := f(a)(z) \quad f(\overline{a}(z)) := \overline{f(a)}(z)$ (an
- The binding restriction ensures that all expressions have a defined value.
- The two-armed conditional if b then P else Q is definable by

Example 12.7

- One-place buffer that outputs non-negative predecessor of stored value:

$$\begin{aligned} B &= in(x).B'(x) \\ B'(x) &= (\text{if } x = 0 \text{ then } \overline{out}(0).B) + (\text{if } x > 0 \text{ then } \overline{out}(x-1).B) \end{aligned}$$

- Input of value "1":

$$\text{(Call)}\ \cfrac{\text{(In)}\ \cfrac{}{in(x).B'(x) \xrightarrow{in(1)} B'(1)}}{B \xrightarrow{in(1)} B'(1)}$$

- Output of predecessor:

$$\text{(Call)}\ \cfrac{\text{(Sum}_2\text{)}\ \cfrac{\text{(If)}\ \cfrac{\text{(Out)}\ \cfrac{}{\overline{out}(1-1).B \xrightarrow{\overline{out}(0)} B}}{\text{if } 1 > 0 \text{ then } \overline{out}(1-1).B \xrightarrow{\overline{out}(0)} B}}{(\text{if } 1 = 0 \text{ then } \overline{out}(0).B) + (\text{if } 1 > 0 \text{ then } \overline{out}(1-1).B) \xrightarrow{\overline{out}(0)} B}}{}$$

Translation of Value-Passing into Pure CCS I

- **To show:** Value-passing process definitions can be represented in pure CCS.
- **Idea:** Each parametrised construct ($a(x)$, $\overline{a}(e)$, $C(e_1, \ldots, e_n)$) corresponds to an indexed family of constructs in pure CCS, one for each possible (combination of) integer value(s).
- Requires extension of pure CCS by infinite choices ("$\sum \ldots$"), restrictions, and process definitions.

Definition 12.8 (Translation of value-passing into pure CCS)

For each $P \in Prc^+$ without free variables, its translated form $\widehat{P} \in Prc$ is given by

$$\widehat{\mathrm{nil}} := \mathrm{nil} \qquad \widehat{\tau.P} := \tau.\widehat{P}$$

$$\widehat{a(x).P} := \sum_{z\in\mathbb{Z}} a_z.\widehat{P[z/x]} \qquad \widehat{\overline{a}(e).P} := \overline{a_z}.\widehat{P} \quad (z \text{ value of } e)$$

$$\widehat{P_1 + P_2} := \widehat{P_1} + \widehat{P_2} \qquad \widehat{P_1 \parallel P_2} := \widehat{P_1} \parallel \widehat{P_2}$$

$$\widehat{P \setminus L} := \widehat{P} \setminus \{a_z \mid a \in L, z \in \mathbb{Z}\} \qquad \widehat{P[f]} := \widehat{P}[\widehat{f}] \quad (\widehat{f}(a_z) := f(a)_z)$$

$$\widehat{\text{if } b \text{ then } P} := \begin{cases} \widehat{P} & \text{if } b \text{ true} \\ \mathrm{nil} & \text{otherwise} \end{cases} \qquad \widehat{C(e_1,\ldots,e_n)} := C_{z_1,\ldots,z_n} \quad (z_i \text{ value of } e_i)$$

Moreover, each defining equation $C(x_1,\ldots,x_n) = P$ of a process identifier is translated into the indexed collection of process definitions

$$\left(C_{z_1,\ldots,z_n} = \widehat{P[z_1/x_1,\ldots,z_n/x_n]} \mid z_1,\ldots,z_n \in \mathbb{Z}\right)$$

Translation of Value-Passing into Pure CCS III

Example 12.9 (cf. Example 12.7)

$$B = in(x).B'(x)$$
$$B'(x) = (\text{if } x = 0 \text{ then } \overline{out}(0).B) + (\text{if } x > 0 \text{ then } \overline{out}(x-1).B)$$

translates to

$B = \sum_{z \in \mathbb{Z}} in_z.B'_z$

$B'_z = P_z + Q_z$ where $P_z := \begin{cases} \overline{out_0}.B & \text{if } z = 0 \\ \text{nil} & \text{otherwise} \end{cases}$ and $Q_z := \begin{cases} \overline{out_{z-1}}.B & \text{if } z > \\ \text{nil} & \text{otherw} \end{cases}$

(for each $z \in \mathbb{Z}$)

Theorem 12.10 (Correctness of translation)

For all $P, P' \in Prc^+$ *and* $\alpha \in Act$,

$$P \xrightarrow{\alpha} P' \iff \widehat{P} \xrightarrow{\widehat{\alpha}} \widehat{P'}$$

where $\widehat{a(z)} := a_z$, $\widehat{\overline{a}(z)} := \overline{a}_z$, *and* $\widehat{\tau} := \tau$.

Proof.

by induction on the structure of P (omitted) □