Concurrency Theory

Winter 2025/26

Lecture 11: Mutually Recursive Equational Systems

Thomas Noll, Peter Thiemann
Programming Languages Group
University of Freiburg

https://proglang.github.io/teaching/25ws/ct.html

Thomas Noll, Peter Thiemann

Winter 2025/26

Syntax of HML with One Recursive Variable

Initially: only one variable (for simplicity; later: mutual recursion)

Definition (Syntax of HML with one variable)

The set HMF_X of Hennessy-Milner formulae with one variable X over a set of actions Act is defined by the following syntax:

$$
\begin{array}{rlll}
F & ::= & X & \text{(variable)} \\
& \mid & \mathrm{tt} & \text{(true)} \\
& \mid & \mathrm{ff} & \text{(false)} \\
& \mid & F_1 \wedge F_2 & \text{(conjunction)} \\
& \mid & F_1 \vee F_2 & \text{(disjunction)} \\
& \mid & \langle \alpha \rangle F & \text{(diamond)} \\
& \mid & [\alpha] F & \text{(box)}
\end{array}
$$

where $\alpha \in Act$.

Semantics of HML with One Recursive Variable

So far: $[\![F]\!] \subseteq S$ for $F \in \mathit{HMF}$ and LTS $(S, \mathit{Act}, \longrightarrow)$.

Now: Semantics of formula depends on states that (are assumed to) satisfy X ("predicate transformer").

Definition (Semantics of HML with one variable)

Let $(S, \mathit{Act}, \longrightarrow)$ be an LTS and $F \in \mathit{HMF}_X$. The semantics of F,

$$[\![F]\!] : 2^S \to 2^S,$$

is defined by

$$\begin{aligned}
[\![X]\!](T) &:= T \\
[\![\mathrm{tt}]\!](T) &:= S \\
[\![\mathrm{ff}]\!](T) &:= \emptyset \\
[\![F_1 \wedge F_2]\!](T) &:= [\![F_1]\!](T) \cap [\![F_2]\!](T) \\
[\![F_1 \vee F_2]\!](T) &:= [\![F_1]\!](T) \cup [\![F_2]\!](T) \\
[\![\langle\alpha\rangle F]\!](T) &:= \langle\cdot\alpha\cdot\rangle([\![F]\!](T)) \\
[\![[\alpha]F]\!](T) &:= [\cdot\alpha\cdot]([\![F]\!](T))
\end{aligned}$$

Applying Fixed-Point Theory to HML with Recursion

Lemma

Let $(S, Act, \longrightarrow)$ *be an LTS and* $F \in HMF_X$*. Then*

(1) $[\![F]\!] : 2^S \to 2^S$ *is monotonic w.r.t.* $(2^S, \subseteq)$

(2) $\mathrm{lfp}([\![F]\!]) = \bigcap\{T \subseteq S \mid [\![F]\!](T) \subseteq T\}$

(3) $\mathrm{gfp}([\![F]\!]) = \bigcup\{T \subseteq S \mid T \subseteq [\![F]\!](T)\}$

If, in addition, S *is finite, then*

(4) $\mathrm{lfp}([\![F]\!]) = [\![F]\!]^m(\emptyset)$ *for some* $m \in \mathbb{N}$

(5) $\mathrm{gfp}([\![F]\!]) = [\![F]\!]^M(S)$ *for some* $M \in \mathbb{N}$

Proof.

(1) by induction on the structure of F (important: HMF_X does not support negation!)

(2) by Corollary 10.10 and Theorem 7.12

Greatest Fixed Points and Invariants I

- Invariants (cf. Example 10.5):
  - $Inv(F) \stackrel{max}{=} F \wedge [Act]Inv(F)$ for $F \in HMF$
  - Claim: $s \models Inv(F)$ if all states reachable from s satisfy F
- Now: formalise argument and prove its correctness (for arbitrary LTSs)
- Let $inv : 2^S \to 2^S : T \mapsto [\![F]\!] \cap [\cdot Act \cdot](T)$ be the corresponding semantic function
- By Lemma 10.11(3), $\text{gfp}(inv) = \bigcup\{T \subseteq S \mid T \subseteq inv(T)\}$
- Direct formulation of invariance property:

$$Inv = \{s \in S \mid \forall w \in Act^*, s' \in S : s \xrightarrow{w} s' \Rightarrow s' \in [\![F]\!]\}$$

Theorem 11.1

For every LTS $(S, Act, \longrightarrow)$, $Inv = \text{gfp}(inv)$ holds.

Greatest Fixed Points and Invariants II

Proof (Theorem 11.1).

Reminder:

- $Inv \stackrel{(*)}{=} \{s \in S \mid \forall w \in Act^*, s' \in S : s \xrightarrow{w} s' \Rightarrow s' \in [\![F]\!]\}$
- $inv : 2^S \to 2^S : T \mapsto [\![F]\!] \cap [\cdot Act \cdot](T)$
- Lemma 10.11(3): $\mathrm{gfp}(inv) = \bigcup\{T \subseteq S \mid T \subseteq inv(T)\}$

"$Inv \subseteq \mathrm{gfp}(inv)$":

According to Lemma 10.11(3), it suffices to show that $Inv \subseteq inv(Inv)$.
So let $s \in Inv$. Thus by $(*)$, for all $w \in Act^*$ and $s' \in S$ such that $s \xrightarrow{w} s'$, $s' \in [\![F]\!]$.

We have to show that $s \in inv(Inv)$, which – by definition of inv – is equivalent to
(1) $s \in [\![F]\!]$ and (2) $s \in [\cdot Act \cdot](Inv)$:

(1) Choose $w := \varepsilon$ in $(*)$.

(2) To show: for all $\alpha \in Act$, $s' \in S$: if $s \xrightarrow{\alpha} s'$, then $s' \in Inv$

Greatest Fixed Points and Invariants II

Proof (Theorem 11.1).

Reminder:

- $Inv \stackrel{(*)}{=} \{s \in S \mid \forall w \in Act^*, s' \in S : s \xrightarrow{w} s' \Rightarrow s' \in [\![F]\!]\}$
- $inv : 2^S \to 2^S : T \mapsto [\![F]\!] \cap [\cdot Act \cdot](T)$
- Lemma 10.11(3): $\mathrm{gfp}(inv) = \bigcup\{T \subseteq S \mid T \subseteq inv(T)\}$

"$\mathrm{gfp}(inv) \subseteq Inv$":

Observation: $\mathrm{gfp}(inv) = inv(\mathrm{gfp}(inv)) \stackrel{(**)}{=} [\![F]\!] \cap [\cdot Act \cdot](\mathrm{gfp}(inv))$.
Let $s \in \mathrm{gfp}(inv)$, $w \in Act^*$ and $s' \in S$ such that $s \xrightarrow{w} s'$.
We show $s' \in [\![F]\!]$ by induction on $|w|$:

(1) $w = \varepsilon$: Here $s = s'$, which implies $s' \in \mathrm{gfp}(inv)$ and thus (by $(**)$) $s' \in [\![F]\!]$.

(2) $w = \alpha w'$: Here $s \xrightarrow{\alpha} s'' \xrightarrow{w'} s'$ for some $s'' \in S$.
Thus, $s'' \in \mathrm{gfp}(inv)$ since $s \in \mathrm{gfp}(inv)$ and by $(**)$.
Therefore, $s' \in [\![F]\!]$ by induction hypothesis for w'.

- Possibilities (cf. Example 10.5):
  - $Pos(F) \overset{min}{=} F \vee \langle Act \rangle Pos(F)$
  - Claim: $s \models Pos(F)$ if a state satisfying F is reachable from s
- Now: formalise argument and prove its correctness (for arbitrary LTSs)
- Let $pos : 2^S \to 2^S : T \mapsto [\![F]\!] \cup \langle \cdot Act \cdot \rangle (T)$ be the corresponding semantic function
- By Lemma 10.11(2), $\mathrm{lfp}(pos) = \bigcap \{T \subseteq S \mid pos(T) \subseteq T\}$
- Direct formulation of possibility property:

$$Pos = \{s \in S \mid \exists w \in Act^*, s' \in [\![F]\!] : s \xrightarrow{w} s'\}$$

Theorem 11.2

For every LTS $(S, Act, \longrightarrow)$, $Pos = \mathrm{lfp}(pos)$ *holds.*

Introducing Several Variables

Sometimes necessary: using more than one variable

Example 11.3

> *"It is always the case that a process can perform an a-labelled transition leading to a state where b-transitions can be executed forever."*

can be specified by

$$Inv(\langle a\rangle Forever(b))$$

where

$$\begin{aligned} Inv(F) &\overset{max}{=} F \wedge [Act]Inv(F) \qquad \text{(cf. Theorem 11.1)} \\ Forever(b) &\overset{max}{=} \langle b\rangle Forever(b) \end{aligned}$$

Syntax of Mutually Recursive Equational Systems

Definition 11.4 (Syntax of mutually recursive equational systems)

Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a set of variables. The set $HMF_\mathcal{X}$ of Hennessy-Milner formulae over $\mathcal{X}$ is defined by the following syntax:

$$
\begin{array}{rlll}
F ::= & X_i & & \text{(variable)} \\
\mid & \mathrm{tt} & & \text{(true)} \\
\mid & \mathrm{ff} & & \text{(false)} \\
\mid & F_1 \wedge F_2 & & \text{(conjunction)} \\
\mid & F_1 \vee F_2 & & \text{(disjunction)} \\
\mid & \langle\alpha\rangle F & & \text{(diamond)} \\
\mid & [\alpha]F & & \text{(box)}
\end{array}
$$

where $i \in [n]$ and $\alpha \in Act$. A mutually recursive equational system has the form

$$(X_i = F_{X_i} \mid 1 \leq i \leq n)$$

where $F_{X_i} \in HMF_\mathcal{X}$ for every $i \in [n]$.

Semantics of Recursive Equational Systems I

As before: Semantics of formula depends on states satisfying the variables.

Definition 11.5 (Semantics of mutually recursive equational systems)

Let $(S, Act, \longrightarrow)$ be an LTS and $E = (X_i = F_{X_i} \mid 1 \leq i \leq n)$ a mutually recursive equational system. The semantics of E, $[\![E]\!] : (2^S)^n \to (2^S)^n$, is defined by

$$[\![E]\!](T_1, \ldots, T_n) := ([\![F_{X_1}]\!](T_1, \ldots, T_n), \ldots, [\![F_{X_n}]\!](T_1, \ldots, T_n))$$

where

$$\begin{aligned}
[\![X_i]\!](T_1, \ldots, T_n) &:= T_i \\
[\![\mathrm{tt}]\!](T_1, \ldots, T_n) &:= S \\
[\![\mathrm{ff}]\!](T_1, \ldots, T_n) &:= \emptyset \\
[\![F_1 \wedge F_2]\!](T_1, \ldots, T_n) &:= [\![F_1]\!](T_1, \ldots, T_n) \cap [\![F_2]\!](T_1, \ldots, T_n) \\
[\![F_1 \vee F_2]\!](T_1, \ldots, T_n) &:= [\![F_1]\!](T_1, \ldots, T_n) \cup [\![F_2]\!](T_1, \ldots, T_n) \\
[\![\langle\alpha\rangle F]\!](T_1, \ldots, T_n) &:= \langle\cdot\alpha\cdot\rangle([\![F]\!](T_1, \ldots, T_n)) \\
[\![[\alpha] F]\!](T_1, \ldots, T_n) &:= [\cdot\alpha\cdot]([\![F]\!](T_1, \ldots, T_n))
\end{aligned}$$

Semantics of Recursive Equational Systems II

Lemma 11.6

Let $(S, Act, \longrightarrow)$ *be a finite LTS and* $E = (X_i = F_{X_i} \mid 1 \leq i \leq n)$ *a mutually recursive equational system. Let* $(D, \sqsubseteq)$ *be given by* $D := (2^S)^n$ *and*

$$(T_1, \ldots, T_n) \sqsubseteq (T'_1, \ldots, T'_n) \qquad \text{iff} \qquad T_i \subseteq T'_i \text{ for every } i \in [n].$$

Then:

(1) $(D, \sqsubseteq)$ *is a complete lattice: if* $\{(T_1^k, \ldots, T_n^k) \mid k \in I\} \subseteq D$ *for some index set* I*, then*

$$\bigsqcup\{(T_1^k, \ldots, T_n^k) \mid k \in I\} = \left(\bigcup_{k \in I} T_1^k, \ldots, \bigcup_{k \in I} T_n^k\right)$$

$$\bigsqcap\{(T_1^k, \ldots, T_n^k) \mid k \in I\} = \left(\bigcap_{k \in I} T_1^k, \ldots, \bigcap_{k \in I} T_n^k\right)$$

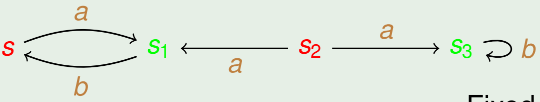
(2) $[\![E]\!]$ *is monotonic w.r.t.* $(D, \sqsubseteq)$

(3) $\mathrm{lfp}([\![E]\!]) = [\![E]\!]^m(\emptyset, \ldots, \emptyset)$ *for some* $m \in \mathbb{N}$

(4) $\mathrm{gfp}([\![E]\!]) = [\![E]\!]^M(S, \ldots, S)$ *for some* $M \in \mathbb{N}$

A Mutually Recursive Specification

Example 11.7



Fixed-point iteration:

- Let $S := \{s, s_1, s_2, s_3\}$ and E:

$$X = \langle a\rangle\mathrm{tt} \wedge [a]Y \wedge [b]\mathrm{ff}$$
$$Y = \langle b\rangle\mathrm{tt} \wedge [b]X \wedge [a]\mathrm{ff}$$

- Interpretation:
  - X: "has no b-successor and ≥ 1 a-successors that all satisfy Y"
  - Y: "has no a-successor and ≥ 1 b-successors that all satisfy X"

$\Rightarrow$ expected: $X = \{s\}$, $Y = \{s_1\}$

$$[\![E]\!]^1(S, S) = (\{s, s_2\} \cap S, \{s_1, s_3\} \cap S) = (\{s, s_2\}, \{s_1, s_3\})$$

$$[\![E]\!]^2(S, S) = (\{s, s_2\} \cap S, \{s_1, s_3\} \cap \{s, s_1, s_2 \ldots = (\{s, s_2\}, \{s_1\})$$

$$[\![E]\!]^3(S, S) = (\{s, s_2\} \cap \{s, s_1, s_3\}, \{s_1, s_3\} \cap \{ \ldots = (\{s\}, \{s_1\})$$

$$[\![E]\!]^4(S, S)$$

Characteristic Formulae

- The Hennessy-Milner theorem asserts that for finitely branching processes, strong bisimilarity and HML-equivalence coincide.
- As a next step, we show that for finite transition systems, the equivalence classes under $\sim$ can be characterised by a system of formulae in HML extended with recursion – one for each state.
- For a finite process P, this HML-formula is called P's characteristic formula as it exactly characterises the $\sim$-equivalence class of P.

The Need for Recursion

Lemma 11.8

There is no recursion-free formula $F \in HMF$ that can characterise the process $A^\omega = a.A^\omega$ up to strong bisimilarity.

Proof.

- Assume $F \in HMF$ with $[\![F]\!] = \{P \in Prc \mid P \sim A^\omega\}$.
- Obviously $a^i \not\sim A^\omega$ for every $i \geq 0$.
- On the other hand, $A^\omega \models F$ implies (by Lemma 9.9) that $a^k \models F$ where k is the modal depth of F.
- Thus, $a^k \sim A^\omega$, which contradicts $a^i \not\sim A^\omega$. □

Lemma (Lemma 9.9)

For every $F \in HMF$, $A^\omega \models F$ iff $a^k \models F$, where k is the modal depth[a] of F.

Characteristic Formulae by Example

- Consider the finite LTS $(S, Act, \longrightarrow)$, and let $\mathcal{X}$ contain (at least) $|S|$ variables.
- Intuitively, X_P is the syntactic symbol for the characteristic formula of process $P \in S$.
- A characteristic formula for P has to describe
  - which actions P can perform,
  - what happens after performing an action, and
  - which actions it cannot perform.

Example 11.9 (Coffee/tea machine; cf. Example 3.13)

$$M = m.M' \qquad M' = \bar{c}.M + \bar{t}.M$$

Observations:

(1) M can perform m and become M'

(2) Performing m, M necessarily becomes M'

(3) M cannot perform any other

Therefore:

$$X_M = \langle m \rangle X_{M'} \wedge [m] X_{M'} \wedge [\{\bar{c}, \bar{t}\}]\mathrm{ff}$$

Similarly:

$$X_{M'} = \langle \bar{c} \rangle X_M \wedge \langle \bar{t} \rangle X_M \wedge [\{\bar{c}, \bar{t}\}] X_M \wedge [m]\mathrm{ff}$$

The General Case

Enabled actions: $P \models \bigwedge_{\{\alpha, P' \mid P \xrightarrow{\alpha} P'\}} \langle\alpha\rangle X_{P'}$

Resulting states: $P \models \bigwedge_{\{\alpha \mid P \xrightarrow{\alpha}\}} [\alpha] \left(\bigvee_{\{P' \mid P \xrightarrow{\alpha} P'\}} X_{P'} \right)$

Disabled actions: $P \models \bigwedge_{\{\alpha \mid P \not\xrightarrow{\alpha}\}} [\alpha]\mathrm{ff}$

can be combined!

Theorem 11.10 (Characteristic Formula) (Ingolfsdottir et al. 1987)

For a finite-state process $P \in Prc$, let the characteristic formula $X_P \in HMF_{\mathcal{X}}$ be defined by:

$$X_P \overset{max}{=} \bigwedge_{\{\alpha, P' \mid P \xrightarrow{\alpha} P'\}} \langle\alpha\rangle X_{P'} \wedge \bigwedge_{\alpha \in Act} [\alpha] \left(\bigvee_{\{P' \mid P \xrightarrow{\alpha} P'\}} X_{P'} \right)$$

(where $\bigwedge_{\emptyset} \ldots := \mathrm{tt}$ and $\bigvee_{\emptyset} \ldots := \mathrm{ff}$). Then, for every $Q \in Prc$: $Q \models X_P$ iff $P \sim Q$.

Proof.

Mixing Least and Greatest Fixed Points I

- **So far:** least/greatest fixed point of overall system
- **But:** too restrictive

Example 11.11

> *"It is possible for the system to reach a state which has a livelock (i.e., an outgoing infinite sequence of internal steps)."*

can be specified by

$$\mathit{Pos}(\mathit{Livelock})$$

where

$$\mathit{Pos}(F) \overset{min}{=} F \vee \langle \mathit{Act} \rangle \mathit{Pos}(F) \qquad \text{(cf. Theorem 11.2)}$$
$$\mathit{Livelock} \overset{max}{=} \langle \tau \rangle \mathit{Livelock}$$

and thus $\mathit{Livelock} \equiv \mathit{Forever}(\tau)$ (cf. Example 11.3).

Mixing Least and Greatest Fixed Points II

Caveat: Arbitrary mixing can entail non-monotonic behaviour!

Example 11.12

$$E : X \overset{min}{=} Y$$
$$Y \overset{max}{=} X$$

Fixed-point iteration:

$$(\bot, \top) = (\emptyset, S) \overset{[\![E]\!]}{\mapsto} (S, \emptyset) \overset{[\![E]\!]}{\mapsto} (\emptyset, S) \overset{[\![E]\!]}{\mapsto} \ldots$$

Solution: Nesting of specifications by partitioning equations into a sequence of blocks such that all equations in one block

- are of same type (either *min* or *max*) and
- use only variables defined in the same or subsequent blocks.

$\Rightarrow$ Bottom-up, block-wise evaluation by fixed-point iteration

Mixing Least and Greatest Fixed Points III

Example 11.13 (cf. Example 11.11)

$$PosLL \overset{min}{=} Livelock \vee \langle Act \rangle PosLL$$
$$Livelock \overset{max}{=} \langle \tau \rangle Livelock$$

$$\begin{tikzcd} s \arrow[r, "a"] & p \arrow[r, "\tau"] \arrow[loop below, "\tau"] & q \arrow[r, "\tau"] & r \end{tikzcd}$$

(1) Greatest fixed-point iteration for $Livelock : T \mapsto \langle \cdot \tau \cdot \rangle (T)$:

$$\top = S = \{s, p, q, r\} \mapsto \{p, q\} \mapsto \{p\} \mapsto \{p\}$$

(2) Least fixed-point iteration for $PosLL : T \mapsto \{p\} \cup \langle \cdot Act \cdot \rangle (T)$:

$$\bot = \emptyset \mapsto \{p\} \mapsto \{s, p\} \mapsto \{s, p\}$$

The Modal μ-Calculus

- Logic that supports free mixing of least and greatest fixed points (with guardedness conditions):
  - originally introduced by D. Kozen: *Results on the Propositional μ-Calculus*, Theoretical Computer Science 27, 1983
  - overview paper by J. Bradfield, C. Stirling: *Modal Logics and mu-Calculi: An Introduction*, Chapter 4 of *Handbook of Process Algebra*, Elsevier, 2001
- HML variants are fragments thereof
- Expressivity increases with alternation of least and greatest fixed points:
  - J.C. Bradfield: *The Modal Mu-Calculus Alternation Hierarchy is Strict*, Theoretical Computer Science 195(2), 1998
- Decidable model-checking problem for finite LTSs (in NP $\cap$ co-NP; linear for HML with one variable)
- Generally undecidable for infinite LTSs and HML with one variable (CCS, Petri nets, ...)
- Overview paper:
  - O. Burkart, D. Caucal, F. Moller, B. Steffen: *Verification on Infinite*