# Concurrency Theory

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Lecture 7*: Fixed Points, Induction, Coinduction

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https://proglang.github.io/teaching/25ws/ct.html

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# Outline of Lecture 7*

# Examples of induction and coinduction

## Inductive situations

- Finite lists, finite trees, natural numbers
- Finite traces of a process (“may” termination)
- Derivation trees with only finitely many inference steps

## Coinductive situations

- Streams and infinite trees
- Infinite traces of a process (non-termination)
- Infinite proof trees / circular / self-similar objects

# Mathematical induction

To prove that a property $P(n)$ holds for all natural numbers $n$:

(1) **Base case:** prove $P(0)$.

(2) **Inductive step:** prove $\forall n.\ P(n) \Rightarrow P(n+1)$.

Then $P(n)$ holds for all $n \in \mathbb{N}$.

Variants are possible, e.g.

- strong induction (assume $P(k)$ for all $k \leq n$),
- different base points,
- induction on other well-founded orders.

# Example of mathematical induction

**Theorem 7*.1**

*For all* $n \geq 1$, $1 + 2 + \cdots + n \;=\; \frac{n(n+1)}{2}$.

**Proof.**

**Base case ($n = 0$).** $0 = \frac{0\cdot(0+1)}{2} = \frac{0}{2} = 0$.

**Induction step.** Assume the formula holds for $n$, i.e. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
Then for $n + 1$:

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

Factor $(n+1)$:

$$\frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2} + 1\right) = (n+1)\frac{n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

Hence the formula holds for $n + 1$. □

# Reminder: Definition by means of rules

Let $U$ be a set (universe).

**Ground rules.** A ground rule on $U$ is a pair $(S, x)$ where:

- $S \subseteq U$ is a (finite) set of premises;
- $x \in U$ is the conclusion.

We usually write:

$$\frac{S}{x}$$

and say: "from all elements of $S$ we may derive $x$".

A rule $(\emptyset, x)$ is an axiom.

# From rules to a monotone operator

Let $R$ be a set of ground rules on $U$.

We associate to $R$ a function

$$F_R : \mathcal{P}(U) \to \mathcal{P}(U)$$

defined by:

$$F_R(X) = \{\, x \in U \mid \exists (S, x) \in R \text{ with } S \subseteq X \,\}.$$

Intuitively, $F_R(X)$ is the set of all conclusions that can be drawn in one step using premises in $X$.

**Fact.** $F_R$ is monotone:

$$X \subseteq Y \;\Rightarrow\; F_R(X) \subseteq F_R(Y).$$

The set inductively defined by $R$ will later be seen as the least fixed point of $F_R$ (we will show this connection with the fixed-point definition of inductive sets).

# Rule induction: finite traces (may termination)

Consider an LTS with a single observable label and write $P\downarrow$ for "$P$ may terminate" (there exists a finite trace from $P$ to a stopped process).

We define $\downarrow$ by the following rules:

$$\frac{P \text{ is stopped}}{P\downarrow}\;(\textsc{Ax}) \qquad\qquad \frac{P \to P' \quad P'\downarrow}{P\downarrow}\;(\textsc{Inf})$$

- The judgement $P\downarrow$ means: $P$ can reach a stopped process in finitely many transitions.
- This is an <u>inductive</u> definition: we only allow <u>finite</u> proof trees from the rules.

# Equivalent readings for $\downarrow$ (1)

Same rules:

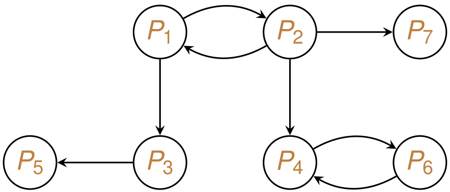
$$\frac{P \text{ is stopped}}{P\downarrow}\ (\textsc{Ax}) \qquad \frac{P \rightarrow P' \quad P'\downarrow}{P\downarrow}\ (\textsc{Inf})$$

**Equivalent readings of $P\downarrow$:**

- The processes $P$ for which there exists a finite proof tree of $P\downarrow$ from the rules.
- The processes $P$ that can reach a stopped process in a finite number of transitions.
- The elements of the least set closed forward under the rules.

# Equivalent readings for $\downarrow$ (2)

Example of finite proof for some processes in an LTS:

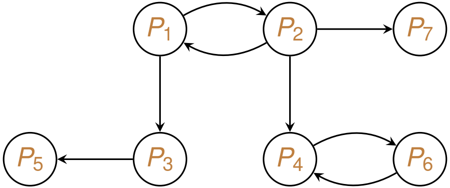



Processes $P_5, P_7$ are stopped. Then we obtain proofs such as:

$$\dfrac{P_3 \longrightarrow P_5 \qquad \dfrac{P_5 \text{ stopped}}{P_5 \downarrow}\,(\text{Ax})}{P_3 \downarrow}\,(\text{Inf}) \qquad \dfrac{P_2 \longrightarrow P_7 \qquad \dfrac{P_7 \text{ stopped}}{P_7 \downarrow}\,(\text{Ax})}{P_2 \downarrow}\,(\text{Inf})$$

and so on, building a finite derivation for $P_3 \downarrow$ and for $P_2 \downarrow$.

# Equivalent readings for $\downarrow$ (3)




- Mark all stopped states as good (they satisfy $\downarrow$).
- Then propagate the mark backwards: whenever $P \rightarrow P'$ and $P'$ is good, then $P$ is good.
- After finitely many steps, we obtain exactly the processes $P$ such that $P \downarrow$.

This is a constructive reading of the least closed set.

# Equivalent readings for $\downarrow$ (4)

Let $S \subseteq U$ be a set of processes.

$S$ is closed forward under the rules for $\downarrow$ if:

- whenever $P$ is stopped, then $P \in S$;
- whenever $P \to P'$ and $P' \in S$, then also $P \in S$.

Then:

- the set of all $P$ such that $P \downarrow$ is the smallest set $S$ that is closed forward under the rules;
- this is exactly the least fixed point of the associated operator $F_R$.

# Using rule induction for finite traces

## Rule induction principle

Let $T$ be a property of processes (a predicate). Suppose:

- $T(P)$ holds for all stopped processes $P$;
- whenever $P \rightarrow P'$ and $T(P')$ holds, then $T(P)$ also holds.

Then $T(P)$ holds for all $P$ such that $P \downarrow$.

- This is the induction principle associated with the inductive definition of $\downarrow$;
- to show that all may-terminating processes enjoy property $T$, it suffices to show that $T$ is closed forward under the rules.

# Example of rule induction for finite traces

Let $f$ be a partial function from processes to values.

$$
\begin{aligned}
f(P) &= 0 && \text{if } P \text{ stopped} \\
f(P) &= \min\{f(P') + 1 \mid \exists P', P \longrightarrow P' \text{ and } f(P') \text{ defined}\} && \text{otherwise}
\end{aligned}
$$

Observe:

- $f(P)$ is defined for every stopped process $P$;
- for every transition $P \rightarrow P'$, if $f(P')$ is defined, then $f(P)$ is defined.

Let $T(P)$ mean “$f(P)$ is defined”. Then $T$ is closed under the rules for $\downarrow$, hence by rule induction:

$$\text{if } P\downarrow \text{ then } f(P) \text{ is defined.}$$

So $f$ is defined on all processes that may terminate.

# Closed sets and forward construction

- Start from the empty set $S_0 = \emptyset$.
- Repeatedly apply the rules forward:

$$S_{i+1} = S_i \cup F_R(S_i).$$

- Take the union $S_\infty = \bigcup_{i \geq 0} S_i$.
- $S_\infty$ is the smallest set closed forward under the rules.
- It coincides with the set of all $P$ such that $P \downarrow$.
- This is another reading of the least fixed point via iterated approximation from below.

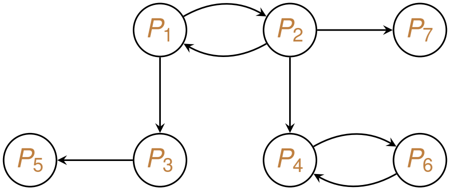
# Rule coinduction: $\omega$-traces (non-termination)

We now define a predicate $P \uparrow_\omega$ meaning: "$P$ has an infinite trace (does not terminate)".

Coinductive rule (indicated by thick line):

$$\frac{P \rightarrow P' \quad P' \uparrow_\omega}{P \uparrow_\omega} \; (\textsc{CoInf})$$

- There is no axiom: only the infinite unfolding of the rule.
- Reading: $P$ has an $\omega$-trace if it can perform a step to some $P'$ that again has an $\omega$-trace.



# Equivalent readings for $\uparrow_\omega$ (1)

Same rule:

$$\frac{P \rightarrow P' \quad P' \uparrow_\omega}{P \uparrow_\omega} .$$

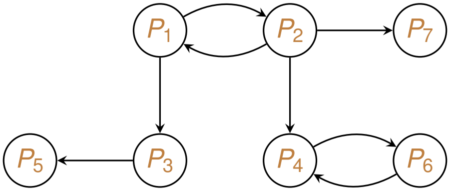
**Equivalent readings of $P \uparrow_\omega$:**

- There exists an infinite proof tree for $P \uparrow_\omega$ using the rule.
- There exists an infinite path

  $$P = P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots$$

  (an $\omega$-trace).
- $P$ belongs to the largest set closed backward under the rule.

# Equivalent readings for $\uparrow_\omega$ (2)




Suppose there is a cycle

$$P_1 \to P_2 \to P_1 .$$

Then we can build an infinite proof:

$$P_1 \uparrow_\omega,\ P_2 \uparrow_\omega,\ P_1 \uparrow_\omega, \ldots$$

This corresponds to the infinite trace

$$P_1, P_2, P_1, P_2, \ldots$$

# Equivalent readings for $\uparrow_\omega$ (3)

Let $S \subseteq U$ be a set of processes.

$S$ is closed backward under the rule if:

$$\text{if } P \in S \text{ then there exists } P' \in S \text{ such that } P \to P'.$$

Then:

- take the set of all $P$ such that $P \uparrow_\omega$ is the largest set $S$ that is closed backward;
- this is the greatest fixed point of the associated operator $F_R$.

# Rule coinduction principle for $\omega$-traces

**Rule coinduction**

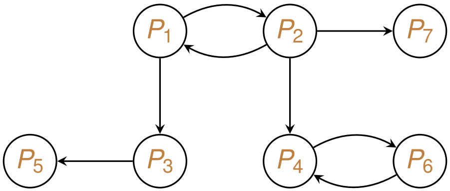
Let $T$ be a property of processes. Suppose:

- whenever $P \in T$ there exists $P' \in T$ such that $P \rightarrow P'$.

Then $T(P)$ implies $P \uparrow_\omega$ for all $P$.

- $T$ is a post-fixed point of the operator associated with the rule.
- By maximality of the greatest fixed point, $T$ is contained in the set of all processes admitting an $\omega$-trace.

# Example of rule coinduction for $\omega$-traces




Consider a set $T$ of processes, for instance $T = \{P_1, P_2\}$. For each $P \in T$ there is a transition to another element of $T$, hence:

- $T$ is closed backward under the rule for $\uparrow_\omega$;
- hence every $P \in T$ satisfies $P \uparrow_\omega$.

Exercise. Would $T = \{P_1, P_2, P_4\}$ be closed backward?

# Equivalent readings for $\uparrow_\omega$ (4)

Still with the rule

$$\frac{P \rightarrow P' \quad P' \uparrow_\omega}{P \uparrow_\omega}.$$

**Equivalent readings:**

- Processes reachable by an infinite proof tree.
- Largest set of processes that is closed backward under the rule.
- (iterative construction) start with the set $Pr$ of all processes;
  repeatedly remove a process $P$ from the set if one of the following conditions applies (i.e.,the backward closure fails):
  - $P$ has no transitions,
  - all transitions from $P$ lead to processes that are not in the set.

# An inductive definition: finite lists over $A$

We want to define the set $L$ of **finite lists** over a set $A$.

$$\frac{}{nil \in L}\ (\textsc{Nil}) \qquad \frac{\ell \in L \quad a \in A}{a :: \ell \in L}\ (\textsc{Cons})$$

## Inductive reading

- $L$ is the smallest set of strings built from $A$ by finitely many applications of these rules.
- Equivalent: start from $\emptyset$ and repeat adding lists using the rules forwards.

## A set $T$ is closed forward

- $nil \in T$
- $\ell \in T$ implies $a :: \ell \in T$, for all $a \in A$

## Inductive proof method for lists

Let $T$ be a predicate on lists. If $T$ is closed forward, then $T$ holds for all lists.

# A coinductive definition: finite and infinite lists

Using the same rules:

$$\frac{}{nil \in L}\ (\textsc{Nil}) \qquad \frac{\ell \in L \quad a \in A}{a :: \ell \in L}\ (\textsc{Cons})$$

**Coinductive reading.**

- $L$ is now the largest set of (finite and infinite) lists closed under the rules.
- A (possibly infinite) list is in $L$ if it can be viewed as being built backwards by repeatedly stripping the head.
- Equivalent: keep removing strings while the backward-closure condition is preserved.

This yields the usual notion of streams over $A$: all finite and infinite lists over $A$.

# A coinductive definition: finite and infinite lists (2)

Using the <u>same</u> rules:

$$\frac{}{nil \in L}\ (\textsc{Nil}) \qquad \frac{\ell \in L \quad a \in A}{a :: \ell \in L}\ (\textsc{Cons})$$

**A set $T$ is closed backward if for all $t \in T$**

- either $t = nil$
- or $t = a :: \ell$, for some $\ell \in T$ and $a \in A$.

**Coinductive proof method**

To prove that $\ell$ is a finite or infinite list, find a set $D$ with $\ell \in D$ and $D$ closed backward.

# A final inductive definition: natural numbers!

We want to define the set $N$ of **natural numbers**.

$$\frac{}{Z \in N}\ (\textsc{Zero}) \qquad \frac{n \in N}{S(n) \in N}\ (\textsc{Succ})$$

## Inductive reading

- $N$ is the smallest set of objects build by finitely many applications of these rules.
- Equivalent: start from $\emptyset$ and repeat adding elements using the rules forwards.

## A set $X$ is closed forward

- $Z \in T$
- $n \in T$ implies $S(n) \in T$

## Inductive proof method for natural numbers

Let $T$ be a predicate on natural numbers. If $T$ is closed forward, then $T$ holds for all natural numbers.