Concurrency Theory

Winter 2025/26

Lecture 5: Game Characterisation and Variants of Strong Bisimulation

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https://proglang.github.io/teaching/25ws/ct.html

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Outline of Lecture 5

1. Recap: Strong Bisimulation
2. Strong Bisimilarity as a Game
3. Simulation Equivalence
4. Summary: Strong (Bi-)Similarity
5. Inadequacy of Strong Bisimilarity
6. Weak Bisimulation

Strong Bisimulation

Definition (Strong bisimulation) (Park 1981, Milner 1989)

A binary relation $\rho \subseteq Prc \times Prc$ is a strong bisimulation whenever for every $(P, Q) \in \rho$ and $\alpha \in Act$:

(1) if $P \xrightarrow{\alpha} P'$, then there exists $Q' \in Prc$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \,\rho\, Q'$, and

(2) if $Q \xrightarrow{\alpha} Q'$, then there exists $P' \in Prc$ such that $P \xrightarrow{\alpha} P'$ and $P' \,\rho\, Q'$.

Note: strong bisimulations are not necessarily equivalences (e.g., $\rho = \emptyset$).

Definition (Strong bisimilarity)

Processes $P, Q \in Prc$ are strongly bisimilar, denoted $P \sim Q$, iff there is a strong bisimulation ρ with $P \,\rho\, Q$.

$$\sim \; = \bigcup \{\rho \subseteq Prc \times Prc \mid \rho \text{ is a strong bisimulation}\}.$$

Relation $\sim$ is called strong bisimilarity.

Properties of Strong Bisimilarity

Lemma (Properties of $\sim$)

(1) $\sim$ *is an equivalence relation (i.e., reflexive, symmetric, and transitive).*

(2) $\sim$ *is the coarsest strong bisimulation.*

Strong Bisimilarity vs. Trace Equivalence

Theorem

$P \sim Q$ implies that P and Q are trace equivalent. The reverse does generally not hold.

Congruence

Theorem (CCS congruence property of $\sim$)

Strong bisimilarity $\sim$ is a CCS congruence, that is, whenever $P, Q \in Prc$ such that $P \sim Q$,

$$\begin{array}{rl} \alpha.P \sim \alpha.Q & \text{for every } \alpha \in Act \\ P + R \sim Q + R & \text{for every } R \in Prc \\ P \parallel R \sim Q \parallel R & \text{for every } R \in Prc \\ P \setminus L \sim Q \setminus L & \text{for every } L \subseteq A \\ P[f] \sim Q[f] & \text{for every } f : A \to A \end{array}$$

Deadlock Sensitivity of Strong Bisimilarity

Definition (Deadlock sensitivity; cf. Definition 3.10)

Relation $\equiv \subseteq Prc \times Prc$ is deadlock sensitive whenever:

$$P \equiv Q \quad \text{implies} \quad (\forall w \in Act^* : P \text{ has a } w\text{-deadlock iff } Q \text{ has a } w\text{-deadlock}) .$$

Theorem

$\sim$ is deadlock sensitive.

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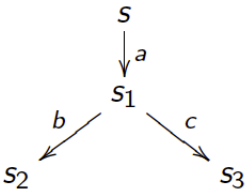

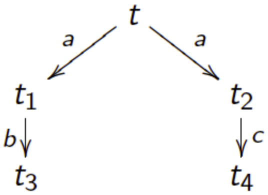

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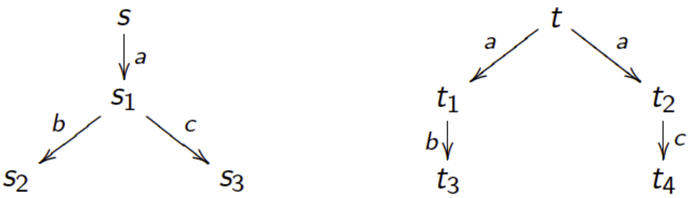

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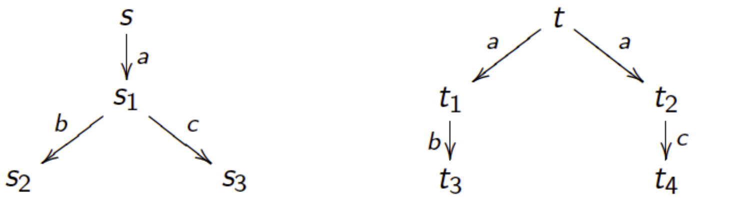
- Enumerate all binary relations and show that none of those containing (s, t) is a strong bisimulation.



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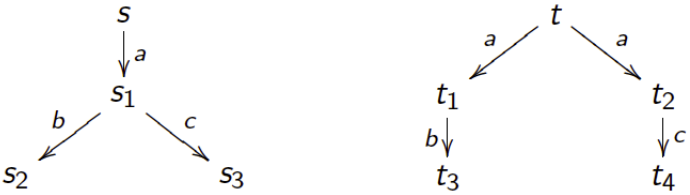

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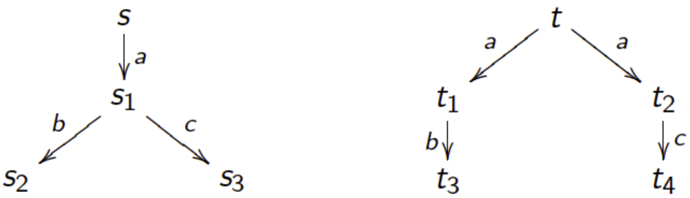

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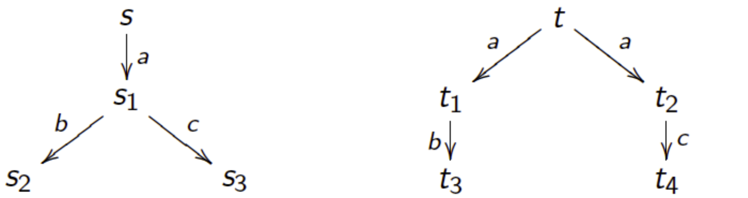

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- Make certain observations which will enable to disqualify many bisimulation candidates in one step.
  - Yields heuristics – how about completeness?
- Use game characterisation of strong bisimilarity.

The Strong Bisimulation Game

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Intuition

The defender wants to show that $s \sim t$ while the attacker aims to prove the opposite.

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In each round, the current configuration (s, t) is changed as follows:

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- defender cannot move if no matching transition available

(2) If the game is played *ad infinitum*, the defender wins.

Example 5.1 (Bisimulation games)

(1) Use the CAAL games feature to show $P \sim Q$ where

$$\begin{array}{rcl} P & = & a.P_1 + a.P_2 \\ P_1 & = & b.P_2 \\ P_2 & = & b.P_2 \end{array} \qquad \begin{array}{rcl} Q & = & a.Q_1 \\ Q_1 & = & b.Q_1 \end{array}$$

Examples

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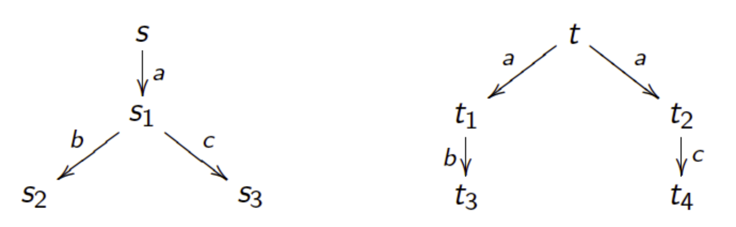

Two winning strategies for attacker in configuration (s, t):

Game Characterisation of Bisimulation

Theorem 5.2 (Game characterisation of bisimulation) (Stirling 1995, Thomas 1993)

(1) $s \sim t$ *iff the defender has a universal winning strategy from configuration* (s, t).

(2) $s \not\sim t$ *iff the attacker has a universal winning strategy from configuration* (s, t).

(By means of a universal winning strategy, a player can always win, regardless of how the other player selects their moves.)

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Thus, a bisimulation game can be used to prove bisimilarity as well as non-bisimilarity.[1] It often provides elegant arguments for $s \not\sim t$.

[1] Later we will present yet another method to check this.

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Definition 5.3 (Strong simulation)

- Relation $\rho \subseteq Prc \times Prc$ is a strong simulation if, whenever $(P, Q) \in \rho$ and $P \xrightarrow{\alpha} P'$, there exists $Q' \in Prc$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \, \rho \, Q'$.

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But: P does not need to be able to match each transition of Q!

Simulation: Example

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Example 5.4

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\begin{tikzcd}
& P \arrow[dl, "a"'] \arrow[dr, "a"] & \\
P_1 \arrow[d, "b"'] & & P_3 \arrow[d, "c"] \\
P_2 & & P_4
\end{tikzcd}
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This yields that:

$$
\begin{array}{lcll}
a.b.\text{nil} + a.c.\text{nil} & \sqsubseteq & a.(b.\text{nil} + c.\text{nil}) & \text{and} \\
a.(b.\text{nil} + c.\text{nil}) & \not\sqsubseteq & a.b.\text{nil} + a.c.\text{nil}. &
\end{array}
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(Note that $P \not\sim Q$.)

Strong Simulation and Bisimilarity

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If $P \sim Q$, then $P \sqsubseteq Q$ and $Q \sqsubseteq P$.

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Example 5.6

$$\begin{array}{ccc} & P & \\ a \swarrow & & \searrow a \\ P_1 & & P_3 \\ \downarrow b & & \\ P_2 & & \end{array} \qquad \begin{array}{c} Q \\ \downarrow a \\ Q_1 \\ \downarrow b \\ Q_2 \end{array}$$

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$P \sqsubseteq Q$ and $Q \sqsubseteq P$, but $P \not\sim Q$

Reason: $\sim$ allows the attacker to switch sides at each step!

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  (2) is an equivalence relation
  (3) is strictly coarser than LTS isomorphism
  (4) is strictly finer than trace equivalence
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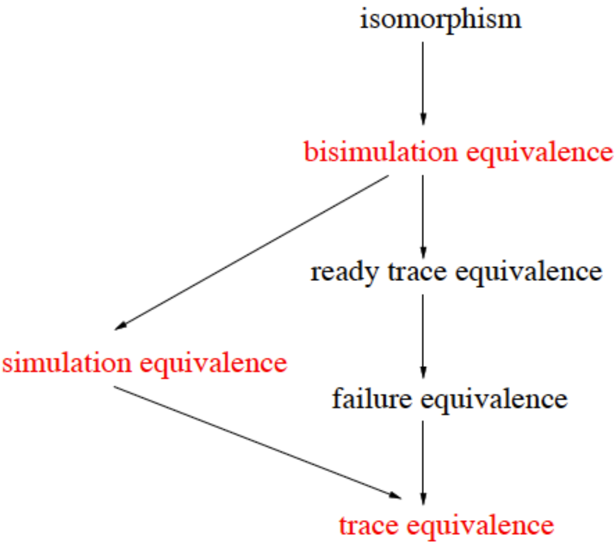
  7. can be checked using a two-player game
- Strong similarity $\sqsubseteq$:
  1. is a one-way strong bisimilarity
  2. bi-directional version (strong simulation equivalence) is strictly coarser than $\sim$

Overview of Some Behavioral Equivalences



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Example 5.7 (Two-place buffers; cf. Example 2.5)

(1) Sequential two-place buffer:

$$\begin{aligned} B_0 &= in.B_1 \\ B_1 &= \overline{out}.B_0 + in.B_2 \\ B_2 &= \overline{out}.B_1 \end{aligned}$$

(2) Parallel two-place buffer:

$$\begin{aligned} B_{\parallel} &= (B[f] \parallel B[g]) \setminus com \\ B &= in.\overline{out}.B \end{aligned}$$

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Observation:

$$\begin{tikzcd} \bullet \arrow[d, shift right, "in"'] \\ \bullet \arrow[u, shift right, "\overline{out}"'] \arrow[d, shift right, "in"'] \\ \bullet \arrow[u, shift right, "\overline{out}"'] \end{tikzcd} \quad \not\sim \quad \begin{tikzcd} & \bullet \arrow[dl, "in"'] & \\ \bullet \arrow[rr, "\tau"] \arrow[dr, "in"'] & & \bullet \arrow[ul, "\overline{out}"'] \\ & \bullet \arrow[ul, "\overline{out}"] & \end{tikzcd}$$

Inadequacy of Strong Bisimilarity

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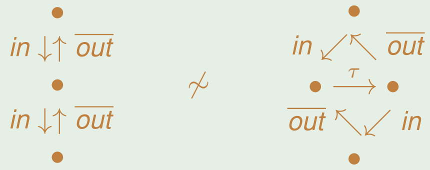
(2) Parallel two-place buffer:

$$\begin{aligned} B_\| &= (B[f] \parallel B[g]) \setminus com \\ B &= in.\overline{out}.B \end{aligned}$$

$(f := [out \mapsto com], g := [in \mapsto com])$

Observation:



Conclusion

- The requirement in $\sim$ to exactly match all actions is often too strong.
- This suggests to weaken this and not insist on exact matching of τ-actions.
- Rationale: τ-actions are special as they are internal and thus unobservable.

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  - thus the result of any communication is unobservable
- Strong bisimilarity treats τ-actions as any other action.
- Can we retain the nice properties of $\sim$ while "abstracting" from τ-actions?

Outline of Lecture 5

1. Recap: Strong Bisimulation
2. Strong Bisimilarity as a Game
3. Simulation Equivalence
4. Summary: Strong (Bi-)Similarity
5. Inadequacy of Strong Bisimilarity
6. Weak Bisimulation

Weak Transition Relation

Definition 5.8 (Weak transition relation)

For $\alpha \in Act$, $\overset{\alpha}{\Longrightarrow} \subseteq Prc \times Prc$ is given by

$$\overset{\alpha}{\Longrightarrow} := \begin{cases} \left(\overset{\tau}{\longrightarrow}\right)^* \circ \overset{\alpha}{\longrightarrow} \circ \left(\overset{\tau}{\longrightarrow}\right)^* & \text{if } \alpha \neq \tau \\ \left(\overset{\tau}{\longrightarrow}\right)^* & \text{if } \alpha = \tau. \end{cases}$$

where $\left(\overset{\tau}{\longrightarrow}\right)^*$ denotes the reflexive and transitive closure of relation $\overset{\tau}{\longrightarrow}$.

Informal meaning

- If $\alpha \neq \tau$, then $P \overset{\alpha}{\Longrightarrow} P'$ means that from P we can get to P' by doing zero or more τ actions, followed by the action α, followed by zero or more τ actions.
- If $\alpha = \tau$, then $P \overset{\alpha}{\Longrightarrow} P'$ means that from P we can reach P' by doing zero or more τ actions.

Weak Bisimulation

Definition 5.9 (Weak bisimulation) (Milner 1989)

A binary relation $\rho \subseteq Prc \times Prc$ is a weak bisimulation whenever for every $(P, Q) \in \rho$ and $\alpha \in Act$ (including $\alpha = \tau$):

(1) if $P \xrightarrow{\alpha} P'$, then there exists $Q' \in Prc$ such that $Q \xRightarrow{\alpha} Q'$ and $P' \mathrel{\rho} Q'$, and

(2) if $Q \xrightarrow{\alpha} Q'$, then there exists $P' \in Prc$ such that $P \xRightarrow{\alpha} P'$ and $P' \mathrel{\rho} Q'$.

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Relation $\approx$ is called weak bisimilarity or observational equivalence.

Explanation

Definition (Weak bisimulation) (Milner 1989)

A binary relation $\rho \subseteq Prc \times Prc$ is a weak bisimulation whenever for every $(P, Q) \in \rho$ and $\alpha \in Act$ (including $\alpha = \tau$):

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Remark

Each clause in the definition of weak bisimulation subsumes two cases:

- $P \xrightarrow{\alpha} P'$ where $\alpha \neq \tau$:
  There exists $Q' \in Prc$ such that $Q \, (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^* \, Q'$ and $P' \,\rho\, Q'$.
- $P \xrightarrow{\tau} P'$:
  There exists $Q' \in Prc$ such that $Q \, (\xrightarrow{\tau})^* \, Q'$ and $P' \,\rho\, Q'$ (where $Q' = Q$ is admissible).

Example 5.11

(1) Let $P = \tau.Q$ with $Q = a.\mathrm{nil}$.

- obviously $P \not\sim Q$; claim: $P \approx Q$
- proof: $\rho = \{(P, Q), (Q, Q), (\mathrm{nil}, \mathrm{nil})\}$ is a weak bisimulation with $P \, \rho \, Q$

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(2) More general: for every $P \in Prc$, $P \approx \tau.P$.

Proof: $\rho = \{(P, \tau.P)\} \cup id_{Prc}$ is a weak bisimulation:

- every transition $P \xrightarrow{\alpha} P'$ can be simulated by $\tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$ (i.e., $\tau.P \xRightarrow{\alpha} P'$) with $P' \, \rho \, P'$ (since $id_{Prc} \subseteq \rho$)
- the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$; it is simulated by $P \xrightarrow{\tau}{}^{0} P$ with $P \, \rho \, P$ (since $id_{Prc} \subseteq \rho$)

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- the only transition of $\tau.P$ is $\tau.P \xrightarrow{\tau} P$; it is simulated by $P \left(\xrightarrow{\tau}\right)^{0} P$ with $P \,\rho\, P$ (since $id_{Prc} \subseteq \rho$)

(3) Sequential and parallel two-place buffer are weakly bisimilar (check with CAAL):

$$
\begin{tikzcd}
P_1 \arrow[d, shift right, "in"'] \\
P_2 \arrow[u, shift right, "\overline{out}"'] \arrow[d, shift right, "in"'] \\
P_3 \arrow[u, shift right, "\overline{out}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Q_1 \arrow[dl, "in"'] & \\
Q_2 \arrow[rr, "\tau"] & & Q_3 \arrow[ul, "\overline{out}"'] \arrow[dl, "in"] \\
 & Q_4 \arrow[ul, "\overline{out}"] &
\end{tikzcd}
$$

$$\rho = \{(P_1, Q_1), (P_2, Q_2), (P_2, Q_3), (P_3, Q_4)\}$$