Concurrency Theory

Winter 2025/26

Lecture 4: Strong Bisimulation

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https://proglang.github.io/teaching/25ws/ct.html

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Outline of Lecture 4

The Wish List for Behavioural Equivalences

(1) Less distinctive than isomorphism: an equivalence should distinguish less processes than LTS isomorphism does, i.e., $\equiv$ should be coarser than LTS isomorphism:

$$LTS(P) \equiv_{iso} LTS(Q) \Rightarrow P \equiv Q.$$

(2) More distinctive than trace equivalence: an equivalence should distinguish more processes than trace equivalence does, i.e., $\equiv$ should be finer than trace equivalence:

$$P \equiv Q \Rightarrow Tr(P) = Tr(Q).$$

(3) Congruence property: the equivalence must be substitutive with respect to all CCS operators (in the following).

(4) Deadlock preservation: equivalent processes should have the same deadlock behaviour, i.e., they can either both deadlock, or both cannot (in the following).

(5) Optional: the coarsest possible equivalence: there should be no less discriminating equivalence satisfying all these requirements.

Trace Equivalence

Definition (Trace language)

For every $P \in Prc$, let $Tr(P) := \{w \in Act^* \mid \text{ex. } P' \in Prc \text{ such that } P \xrightarrow{w} P'\}$ be the trace language of P (where $\xrightarrow{w} := \xrightarrow{\alpha_1} \circ \ldots \circ \xrightarrow{\alpha_n}$ for $w = \alpha_1 \ldots \alpha_n$).

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- Solution: consider finer behavioural equivalences $\equiv$ such that
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- Our (serious) attempt today: Milner's strong bisimulation.

Robin Milner (1934–2010)

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Observation

In order for a behavioural equivalence to be deadlock sensitive, it has to take the branching structure of processes into account.

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This is achieved by an equivalence that is defined according to the following scheme:

Bisimulation scheme

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Three variants will be considered in this course:

(1) Strong bisimulation: ignore the special role of τ-actions
(2) Weak bisimulation: treat τ-actions as invisible
(3) Simulation relations: unidirectional versions of bisimulation

Strong Bisimulation I

Definition 4.1 (Strong bisimulation) (Park 1981, Milner 1989)

A binary relation $\rho \subseteq Prc \times Prc$ is a strong bisimulation if for every $(P, Q) \in \rho$ and $\alpha \in Act$:

(1) if $P \xrightarrow{\alpha} P'$, then there exists $Q' \in Prc$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \, \rho \, Q'$, and

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Relation $\sim$ is called strong bisimilarity.

Strong Bisimulation II

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P' \\ \rho & & \\ Q & & \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} P & \xrightarrow{\alpha} & P' \\ \rho & & \rho \\ Q & \xrightarrow{\alpha} & Q' \end{array}$$

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Examples

Definition 4.3 (Strong bisimulation — recall) (Park 1981, Milner 1989)

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Example 4.4 (A first example)

Claim: $P \sim Q$ where

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A Counterexample

Example 4.6 (Vending machines; cf. Example 3.13)

Show $CTM \not\sim CTM'$ where
$$\begin{aligned} CTM &= coin.\left(\overline{coffee}.CTM + \overline{tea}.CTM\right) \\ CTM' &= coin.\overline{coffee}.CTM' + coin.\overline{tea}.CTM'. \end{aligned}$$

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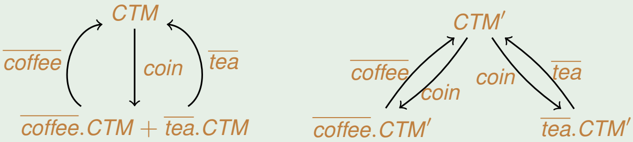

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$$\begin{tikzcd}
CTM \arrow[d, "coin"] \\
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\end{tikzcd}
\qquad
\begin{tikzcd}
 & CTM' \arrow[dl, bend left, "coin"] \arrow[dr, bend right, "coin"'] & \\
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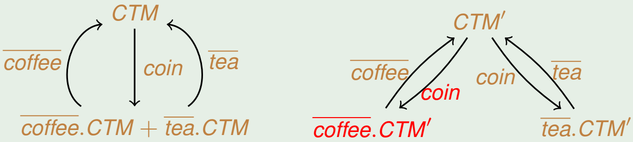

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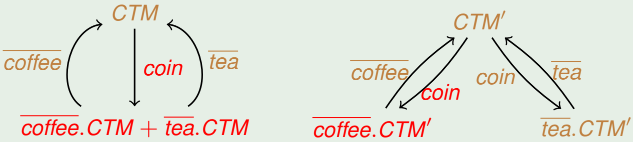

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- First CTM' chooses the left $coin$-transition.
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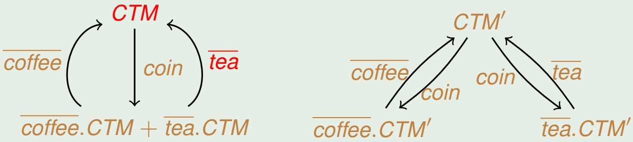

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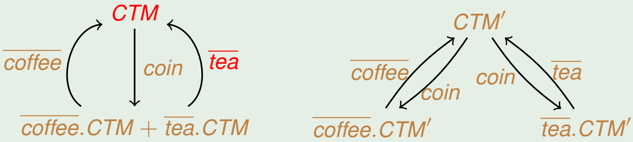

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- CTM proceeds by selecting the $\overline{tea}$-transition.
- But CTM' cannot react to this step. ↯

(Verify using CAAL)

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Proof.

(1) $\sim$ is an equivalence relation:

- Reflexivity:

$$\mathrm{id}_{Prc} := \{(P, P) \mid P \in Prc\}$$

  is obviously a strong bisimulation.
  Since $\mathrm{id}_{Prc} \subseteq \sim$ by Definition 4.2, $\sim$ is reflexive.

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Proof.

(1) $\sim$ is an equivalence relation:

- Symmetry: (**Caveat:** not every strong bisimulation is symmetric; cf. Example 4.4.)
  But if ρ is a strong bisimulation, then so is its inverse

$$\rho^{-1} := \{(Q, P) \mid P\rho Q\}$$

  (due to symmetry in Definition 4.3). Therefore, $\sim$ is symmetric by Definition 4.2.

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Proof.

(1) $\sim$ is an equivalence relation:

- Transitivity: (**Caveat:** not every strong bisimulation is transitive.)
  But if ρ and σ are strong bisimulations, then so is their composition $\rho \circ \sigma := \{(P, R) \mid \exists Q : P\rho Q, Q\sigma R\}$.
  Proof: $P\ (\rho \circ \sigma)\ R$ and $P \xrightarrow{\alpha} P'$
  $\Rightarrow \exists Q : P\rho Q, Q\sigma R$ and $P \xrightarrow{\alpha} P'$ (def. $\circ$)
  $\Rightarrow \exists Q, Q' : Q\sigma R, Q \xrightarrow{\alpha} Q'$ and $P' \rho Q'$ (ρ strong bisimulation)
  $\Rightarrow \exists Q', R' : P' \rho Q', R \xrightarrow{\alpha} R'$ and $Q' \sigma R'$ (σ strong bisimulation)
  $\Rightarrow \exists R' : R \xrightarrow{\alpha} R'$ and $P'\ (\rho \circ \sigma)\ R'$ (def. $\circ$)
  (analogously for assumption $R \xrightarrow{\alpha} R'$)
  Therefore, $\sim$ is transitive by Definition 4.2.

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Proof.

(2) $\sim$ is the coarsest strong bisimulation:
According to Definition 4.2, it suffices to show that strong bisimulations are closed under union, i.e., whenever ρ, σ are bisimulations, then so is $\rho \cup \sigma$. This immediately follows by case distinction. $\square$

Outline of Lecture 4

Lemma 4.8 (Bisimulation on paths)

Whenever we have:

$$
\begin{array}{ccccccccc}
P_0 & \xrightarrow{\alpha_1} & P_1 & \xrightarrow{\alpha_2} & P_2 & \xrightarrow{\alpha_3} & P_3 & \xrightarrow{\alpha_4} & P_4 \ldots\ldots \\
\rho & & & & & & & & \\
Q_0 & & & & & & & &
\end{array}
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this can be completed to

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Proof.

by induction on the length of the path $\square$

Strong Bisimilarity vs. Trace Equivalence

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Consider the other direction:

- Take $P = a.P_1$ with $P_1 = b.\mathrm{nil} + c.\mathrm{nil}$ and $Q = a.b.nil + a.c.\mathrm{nil}$.

Strong Bisimilarity vs. Trace Equivalence

Theorem 4.9

$P \sim Q$ implies that P and Q are trace equivalent. The reverse does generally not hold.

Proof.

The implication from left to right follows from Lemma 4.8.

Consider the other direction:

- Take $P = a.P_1$ with $P_1 = b.\mathrm{nil} + c.\mathrm{nil}$ and $Q = a.b.nil + a.c.\mathrm{nil}$.
- Then: $Tr(P) = \{\epsilon, a, ab, ac\} = Tr(Q)$.
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- Then: $Tr(P) = \{\epsilon, a, ab, ac\} = Tr(Q)$.
- Thus, P and Q are trace equivalent.
- But: $P \not\sim Q$, as there is no state in the LTS of Q that is bisimilar to P_1 (cf. Example 4.6).
- Why? Since no state in Q can perform both b and c.

$\square$

Definition 4.10 (Determinism)

$P \in Prc$ is deterministic whenever for every of its reachable states R it holds:

$$\left(R \xrightarrow{\alpha} R' \text{ and } R \xrightarrow{\alpha} R''\right) \quad \text{implies} \quad R' = R''.$$

Deterministic Transition Systems I

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Theorem 4.11 (Determinism implies coincidence of $\sim$ and trace equiv.) (Park 1981)

For deterministic P and Q: $P \sim Q$ iff $Tr(P) = Tr(Q)$.

Deterministic Transition Systems II

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By Theorem 4.9, it remains to prove that $Tr(P) = Tr(Q)$ implies $P \sim Q$.

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To this end, we show that

$$\rho := \{(R, S) \mid P \longrightarrow^* R, Q \longrightarrow^* S, Tr(R) = Tr(S)\}$$

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- Hence ex. $S' \in Prc$ with $S \xrightarrow{\alpha} S'$.
- Again by determinism, $\{w \in Tr(S) \mid w = \alpha \ldots\} = \alpha \cdot Tr(S')$.
- Altogether, $Tr(R') = Tr(S')$ and thus $R' \rho S'$, which completes the proof. $\square$

Outline of Lecture 4

Congruence I

Theorem 4.12 (CCS congruence property of $\sim$)

Strong bisimilarity $\sim$ *is a CCS congruence, that is, whenever* $P, Q \in Prc$ *such that* $P \sim Q$*,*

$$\begin{array}{rl} \alpha.P \sim \alpha.Q & \text{for every } \alpha \in Act \\ P + R \sim Q + R & \text{for every } R \in Prc \\ P \parallel R \sim Q \parallel R & \text{for every } R \in Prc \\ P \setminus L \sim Q \setminus L & \text{for every } L \subseteq A \\ P[f] \sim Q[f] & \text{for every } f : A \to A \end{array}$$

Congruence II

Proof.

We only consider parallel composition and prove $P \parallel R \sim Q \parallel R$ by showing that

$$\rho := \{(P' \parallel R', Q' \parallel R') \mid P' \sim Q', R \longrightarrow^* R'\}$$

is a strong bisimulation.

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To this aim, let $(P' \parallel R')\ \rho\ (Q' \parallel R')$.

- If $P' \parallel R' \xrightarrow{\alpha} S'$, the following cases are possible:

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To this aim, let $(P' \parallel R')\ \rho\ (Q' \parallel R')$.

- If $P' \parallel R' \xrightarrow{\alpha} S'$, the following cases are possible:
  (1) $P' \xrightarrow{\alpha} P''$ and $S' = P'' \parallel R'$:
  Since $P' \sim Q'$, there ex. Q'' such that $Q' \xrightarrow{\alpha} Q''$ and $P'' \sim Q''$.
  Thus, $Q' \parallel R' \xrightarrow{\alpha} Q'' \parallel R'$ and $S'\ \rho\ (Q'' \parallel R')$.

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  (2) $R' \xrightarrow{\alpha} R''$ and $S' = P' \parallel R''$:
  Here $Q' \parallel R' \xrightarrow{\alpha} Q' \parallel R''$ and $S' \, \rho \, (Q' \parallel R'')$.

Proof.

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  Here $Q' \parallel R' \xrightarrow{\alpha} Q' \parallel R''$ and $S'\ \rho\ (Q' \parallel R'')$.
  (3) $\alpha = \tau$, $P' \xrightarrow{\lambda} P''$, $R' \xrightarrow{\overline{\lambda}} R''$ (for some $\lambda \in A \cup \overline{A}$) and $S' = P'' \parallel R''$:
  combination of (1) and (2).

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  combination of (1) and (2).
- $Q' \parallel R' \xrightarrow{\alpha} T'$: analogous

$\square$

Outline of Lecture 4

Deadlock Sensitivity of Strong Bisimilarity

Definition (Deadlock; cf. Definition 3.8)

Let $P, Q \in Prc$ and $w \in Act^*$ such that

$$P \xrightarrow{w} Q \quad \text{and} \quad Q \not\longrightarrow .$$

Then Q is called a *w-deadlock* of P.

Definition (Deadlock sensitivity; cf. Definition 3.10)

Relation $\equiv \subseteq Prc \times Prc$ is *deadlock sensitive* whenever:

$$P \equiv Q \quad \text{implies} \quad (\forall w \in Act^* : P \text{ has a } w\text{-deadlock iff } Q \text{ has a } w\text{-deadlock}) .$$

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Let $P \sim Q$.

- We assume that, for some $w \in Act^*$, P has a w-deadlock but Q does not (or vice versa).

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- Thus, there exists $P' \in Prc$ such that $P \xrightarrow{w} P'$ and $P' \not\longrightarrow$.
- Moreover, for all $Q' \in Prc$ with $Q \xrightarrow{w} Q'$ there exist $\alpha \in Act$ and $Q'' \in Prc$ such that $Q' \xrightarrow{\alpha} Q''$.

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- Thus, there exists $P' \in Prc$ such that $P \xrightarrow{w} P'$ and $P' \not\longrightarrow$.
- Moreover, for all $Q' \in Prc$ with $Q \xrightarrow{w} Q'$ there exist $\alpha \in Act$ and $Q'' \in Prc$ such that $Q' \xrightarrow{\alpha} Q''$.
- For $P \xrightarrow{w} P'$, Lemma 4.8 (bisimulation on paths) yields Q' with $Q \xrightarrow{w} Q'$ and $P' \sim Q'$.
- Thus $P' \not\longrightarrow$ and $Q' \xrightarrow{\alpha} Q''$ cannot hold at the same time. $\lightning$

$\square$

Outline of Lecture 4

Example 4.14 (An n-ary semaphore)

S_i^n stands for a semaphore for n identical, exclusive resources i of which are taken:

$$\begin{aligned} S_0^n &= get.S_1^n \\ S_i^n &= get.S_{i+1}^n + put.S_{i-1}^n \quad \text{for } 0 < i < n \\ S_n^n &= put.S_{n-1}^n \end{aligned}$$

Semaphores I

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This process is strongly bisimilar to n parallel binary semaphores:

Lemma 4.15

For every $n \in \mathbb{N}_+$, *we have:* $S_0^n \sim \underbrace{S_0^1 \parallel \cdots \parallel S_0^1}_{n \text{ times}}$.

Semaphores II

Lemma

For every $n \in \mathbb{N}_+$*, we have:* $S_0^n \sim \underbrace{S_0^1 \parallel \cdots \parallel S_0^1}_{n \text{ times}}$.

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Proof.

Consider the following binary relation where $i_1, \ldots, i_n \in \{0, 1\}$:

$$\rho = \left\{ \left(S_k^n, S_{i_1}^1 \parallel \cdots \parallel S_{i_n}^1\right) \;\middle|\; \sum_{j=1}^{n} i_j = k \right\}$$

Semaphores II

Lemma

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Then: ρ is a strong bisimulation and $(S_0^n, \underbrace{S_0^1 \parallel \cdots \parallel S_0^1}_{n \text{ times}}) \in \rho$. □